Higher Topics

Unit 1

1. The straight line

Distance formula
Mid-point formula
Converse of Pythagoras
Angles in a triangle – largest angle opposite longest side
Gradient formula
m = tan θ
parallel lines have the same gradient
lines with the same gradient are parallel
gradient of line parallel to x-axis is zero
gradient of line parallel to y-axis is undefined
positive and negative gradients
Finding angle between line and positive direction of x-axis
Using gradients to show points are collinear
Perpendicular lines $m_1 = -\frac{1}{m_2}$
Right angled triangles and perpendicular gradients
Does a point lie inside, on or outside of a circle
Equation of a straight line: gradient m and y-intercept c
Linear equation ax + by + c = 0
Line with gradient m through the point (a, b): y – b = m(x-a)
Median of a triangle
Altitude of a triangle
Finding equation of a median
Finding equation of an altitude
Intersecting lines – simultaneous equations
Bearings – shortest distance from a line to a point is perpendicular from point to line

2.1 Composite and Inverse Functions

Definition of a function
Notation
Domain and range
Function Definition– For every x in domain there is only 1 value of f in the range
Find formula for g(2x) and g(x+1)
For what values is a function undefined
Tricks with g(x) and g(1/x) and g(x/(1+x))
Composite functions f(g(x)) and g(f(x))
Functions and inverses – one to one mapping
Notation for an inverse f^{-1}
Checking if an inverse exists from a graph – the ruler test
Calculating inverse functions – y = 2x + 5
Reflection in the line y = x

2.2 Algebraic Functions and Graphs

Completing the square – simple, negative x^2 and non unity x^2
Maximum and minimum values
Value of x for which it occurs
Sketching the graph by completing the square
    Find max or min and value of x at which it occurs

Find roots (zeros) – (put y = 0)
Find y intercept (put x = 0)
Can find turning point from axis of symmetry
Sketching graphs of related functions
y = f(x)   y = -f(x)   y = f(-x)
y = f(x + a)   y = f(x – a)
y = f(x) + a   y = f(x) – a
y = a f(x)   y = f(ax)
Marking the image points
The exponential function and its graph
$f(x) = a^x$ where a > 0
when a > 1 then increasing function
when 0 < a < 1 then a decreasing function
$y = a^x$ always goes through (0, 1) because $a^0 = 1$
Look for the point x = 1 on the graph
then y = a since $a^1 = a$
The logarithmic function and its graph
$y = a^x \Rightarrow x = \log_a y$
Look at reflection in the line y = x
Two special logarithms
$\log_a 1 = 0$ log 1 to any base = 0
$\log_a a = 1$ log of a number to that base is 1
Sketching graphs of logarithmic functions
Use two special points when x = 1 then y = 0 and when x = a then y = 1
Finding the equation of the graph from points on the garph
Finding the values of a and b in the equation $y = a\log_4(x+b)$ from the graph
Use simultaneous equations

2.3 Trigonometric Functions and Graphs

Radian measure
Changing degrees to radians
Changing radians to degrees
checking your calculator
Exact values Table using surds
Angles of all sizes in radian measure
New definition for sin, cos and tan
ASTC
Expressing π as 4π/4, 6π/6 etc
Sketching trig graphs: y = a sin nx and y = a cos nx
a gives max and min values
n gives number of cycles in 360°
Period of graph is 2π/n
Finding co-ordinates of max turning point for y = 3 sin(x - π/3)
Solving trigonometric equations
2 sinx = 1
√2 cos θ + 1 = 0
sin 3x = -1
$2 \sin^2 x = 1$ (4 solutions)
$4 \sin^2 x + 11 \sin x + 6 = 0$ (quadratic equation in sin x)
$\sin^2 x – \cos^2 x = 1$ (using $\sin^2 x + \cos^2 x = 1$)
sin (2x + π/4) = 1

3.1 Introduction to Differentiation

Definition of the derivative
Meaning of the derivative
The limit formula
First Principles
Rules for differentiation

$f(x) = x^n$	$f'(x) = nx^{n-1}$	
$f(x) = x$	$f'(x) = 1$	
$f(x) = cg(x)$	$f'(x) = cg'(x)$	where c is a constant
$f(x) = c$	$f'(x) = 0$	where c is a constant
$f(x) = g(x) + h(x)$		$f'(x) = g'(x) + h'(x)$

variables may be denoted by letters other than x, such as u, v, s, t, etc.

Methods (templates)

simple powers of x	x^5
simple powers of x with a multiplier	$3x^4$
powers of x with 2 terms	$x^2 + x$
powers of x with 2 terms and multiplier	$5x^2 + 3x$
Trinomials with and without multipliers	$\frac{1}{2}x^2 - x + 3$
Brackets (multiply out brackets first)	$(x + 2)^2$
More involved brackets	$(2x - 4)^2$
Pairs of brackets	$(2x + 3)(x - 1)$
Polynomial expressions	$2y^3 - 3t^2 + 5t - 4$
Vary the variables	u, t, v, s etc.

Evaluating $f'(a)$ given a e.g. $f'(0)$ $f'(-1)$ etc

Reminder of what the derivative is: The gradient function
What does the gradient function mean
The gradient (steepness) of the curve at that point
The gradient of the tangent to the curve at that point.

Find the gradient of the curve f(x) at x = a
Linking the rate of change to the gradient
Finding the rate of change of f with respect to x at x = 2
or the rate of change of x^2 at x = 3

Reminder of rules of indices:

$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad \left(a^m\right)^n = a^{mn}$$

$$a^0 = 1$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^{\frac{1}{n}} = \sqrt[n]{a} \qquad \textit{in general} \qquad a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Expressing fractions and roots in straight line index form
Writing indices in positive index form
Simplification

Evaluating indexed expressions e.g. $t^{\frac{1}{3}}$ *when* $t = -8$

Multiplying out brackets with fractions and indices e.g. $y^{\frac{1}{3}}\left(y^{\frac{2}{3}}-y^{-\frac{1}{3}}\right)$ *or* $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

Rules of differentiation for negative and fractional indices

$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = cg(x)$	$f'(x) = cg'(x)$
$f(x) = c$	$f'(x) = 0$
$f(x) = g(x) + h(x)$	$f'(x) = g'(x) + h'(x)$

Methods (templates)

simple negative powers of x	x^{-2}
simple fractional powers of x	$x^{3/2}$
simple negative fractional powers of x	$x^{-1/2}$
when in fraction form of a power	$\frac{1}{x^3}$
when in fraction form of a root	$\frac{1}{\sqrt{x}}$
more involved root/power form	$\frac{1}{\sqrt[3]{x^4}}$
Dealing with constants – multipliers	$\frac{3}{x}$
Dealing with constants – divisors	$\frac{1}{2x}$
Dealing with multipliers and divisors	$\frac{3}{2\sqrt{x}}$ *or* $\frac{2}{5x^2}$
Sums of expressions	$1+\frac{1}{\sqrt{x^3}}-x^2$

Dealing with quotients $\frac{x^4+2x^2+3}{x}$ $\frac{(x+2)^2}{x^2}$ *or* $\frac{(x-3)(x+2)}{x^3}$

Dealing with brackets and fractions $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$

Brackets with a fractional index multiplier $x^{\frac{3}{2}}\left(x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)$ or $\sqrt[3]{x}\left(1+\frac{1}{\sqrt[3]{x}}\right)$

Leibniz notation dy/dx

Formal proof
Using dy/dx as the notation for differentiation

The form $\frac{d}{dx}\left(2x^2+1\right)$ also d/du ds/dt etc.

Sketching graphs of derived functions
Look for turning points
mark as zero on the axes
Look either side of t.p. for gradient – mark above or below axes
join up points to obtain derived function

Given the derived function – sketch a possible function it could have come from.

3.2 Using Differentiation

Increasing functions
stationary points
stationary values
maximum and minimum turning points
points of inflexion
Look at gradient pattern around these points

If (a, f(a)) is a stationary point, then f′(a) = 0,
so a is a root of the equation f′(x) = 0
and f(a) is a stationary value of a.

The nature of a stationary point can be determined by finding the sign of f′(x) to the left and right of x = a.

Notation SP, SV, TP, PI as acceptable abbreviations.

Finding the stationary point of a parabola and determining its nature
Differentiate
For a SP, dy/dx = 0
Solve to find x co-ordinate,
Substitute in y (or f(x) to find y value
Determine its nature with table of signs
Finding the interval on which f(x) is deceasing or increasing
Curve sketching
Determine:
Points of intersection with x and y axes
Stationary points and their natures
Behaviour of y for large positive and negative x
Any other useful points on the graph
Maximum and minimum values on a closed interval
These can occur at end points or at stationary points in the interval
Closed intervals [4, 3] includes end points $4 \leq x \leq 3$
Open Interval (4, 3) does not include the end points $4 < x < 3$
Problem solving – Optimisation
Finding an expression involving 2 variables
Eliminating a variable using a constraint (given)
Forming a single independent variable expression
Differentiation
Find SP
Find SV (if required)

- the y co-ordinate or the value of f(x) from original expression
Determine its nature using table of signs
(or second derivative)
Answer any further requirements of question by calculations
Some examples here would be useful

Rates of Change
Rate of increase, Rate of decrease
rate of decrease requires a ***negative*** sign
Review direct proportion $y = kx$ and inverse proportion $y = k/x$
Forming expressions and obtaining the rate of increase/decrease
Evaluating the rate of change (increase/decrease)

Velocity and acceleration
Velocity = rate of change of displacement wrt time $v = dx/dt$
Acceleration = rate of change of velocity wrt time $a = dv/dt$
Calculation of acceleration, velocity by differentiation
Evaluating at particular values
Height problems
Max height is reached when velocity $dh/dt=0$

4. Sequences

Rules for sequences
Two ways of defining
Formula for nth term
One term and a method for finding each of the following terms
Examples of nth term
Given formula for nth term calculate terms in sequence
Finding nth terms
Look at difference for a linear sequence
Recurrence Relation
Given first term and a formula
$u_{n+1} = u_n + 3 \quad u_1 = 7$
Called a recurrence relation because successive terms can be found by recurring use of it.
Find terms from a recurrence relation
Find specific terms from recurrence relation e.g. given relation and u_1 find u_0
Form a recurrence relation from a statement
Linear recurrence relations
are of the form $u_{n+1} = mu_n + c$ where m and c are constants
Arithmetic series: m = 1 - difference between successive terms is constant
Examples: Deposits and simple payments, simple interest,
pay rises – simple increment
Geometric Sequences: c = 0 - ratio of successive terms is constant
Examples – appreciation, depreciation, compound interest

Finite and infinite Sequences
n tending to infinity notation $n \to \infty$ and $1/n \to 0$
Look for terms of form 1/n to see what happens as $n \to \infty$
if $1/n \to 0$ then $1/n^2 \to 0$ even faster – show example
both with numbers and graphically
Convergent – tends to a limit
Finding limits of sequences – some examples and tricks

$$u_n = \frac{2n+1}{n} \quad and \quad u_n = \frac{n}{n+1}$$

Finding the sum to infinity of a geometric sequence

Proof

Sum to n terms

Let n tend to infinity

Stipulation $|r| < 1$ i.e. $-1 < r < 1$

The linear recurrence relation $u_{n+1} = mu_n + c$

with $m \neq 1$ and $c \neq 0$

Example of formation

Calculating first few terms

Looking for a trend

Limit

Limit statement if $-1 < m < 1$ then the sequence will tend to a limit L.

Derive the limit formula – numerically

then using letters

Finding whether a limit exists

Forming a recurrence relation

Understanding the question with more complicate examples

Answering the outcome – what happens in the long term

Settles out at around

Do not say drops down to or rises up to unless you know the initial value

Examples of why this is so

Also advisable to check first few terms of sequence to make sure nothing unusual is happening – do this graphically

Working with recurrence relations

A sequence is given: $u_{n+1} = au_n + b$ where a and b are constants

given u_1 and u_2, find a and b (simultaneous equations)

Can also word it in the form of find m and c

Find limit of sequence

Find difference between a given term and limit.

Unit 2

1.1 Polynomials

Define a polynomial
Define degree
Demonstrate the nested scheme
    Proof
    Method
Finding the value of the polynomial when x = some value
Division of polynomials
    Synthetic division
    Divisor, Quotient, remainder
    Meaning of quotient as coefficients
    Be careful writing down coefficients of polynomial
        – watch out for missing powers
    Finding the divisor
        Divisor is always of the form $x - h$
        so, dividing by $x + 3$ means a divisor of $(x - (-3))$
        Synthetic division is with –3
    Dividing by $(ax + b)$
        this is of the form $a(x + b/a) = a(x - (-b/a))$
        so, synthetic division is with $-b/a$
        Then the quotient also has to be divided by a,
            but not the remainder
                – show this with numerical examples first
Remainder Theorem
    Extend proof from quadratic to a polynomial of any degree
    When a polynomial $f(x)$ is divided by $x - h$ the remainder is $f(h)$

Factor Theorem
    If $f(h) = 0$ then $x-h$ is a factor of $f(x)$

Finding factors: look at constant terms what are the factors
    if last term is 6 then factors of 6 are ±1, ±2, ±3, ±6
    Try each in turn
Use $f(h)$ if just trying to find a factor
Use synthetic division if you want to find the quotient as well.
Once you have found a factor, you can complete the factorisation
    repeat as many times as required, until you have factorised it
    or got down to a quadratic term which will factorise.
Given $(x + 1)$ is a factor of $f(x) = \ldots\ldots + a$ find a or variations
Can have two variables and two factors given
Having got a factor and a quadratic expression factorise fully
Solving polynomial equations
    Find a factor
    Factorise quotient
    Solve equation
Showing equation has only one real root – show only one factor and
    remaining quadratic expression will not factorise – or no solution
Solution of the equation $f(x) = 0$ is where the curve cuts the x axis
Solution of the equation $V = f(x)$ and $V = 10$ (say) is solving $f(x) = 10$
    or $f(x) - 10 = 0$
Approximate roots – iteration
    Taking a guess – trial and error

1.2 Quadratic Theory

Define quadratic function: $f(x) = ax^2 + bx + c$
Define quadratic expression: $ax^2 + bx + c$
Define a quadratic equation: $ax^2 + bx + c = 0$
Putting a quadratic expression into standard form $ax^2 + bx + c$
Solving quadratic equations
graphically
Factorising
when it cannot be factorised
completing the square
The quadratic formula
derivation
Using the quadratic formula
a method
Solving using:
factors
completing the square#
the quadratic formula
The discriminant

$b^2 - 4ac$	> 0	real and distinct root
	$= 0$	equal roots
	< 0	no real roots

graphical illustraion
$b^2 - 4ac$ discriminates between the roots
which is why it is called the discriminant
For what value of p does the equation $x^2 - 2x + p = 0$ have
equal roots, real and distinct, no real roots
Find m given that the equation $x^2(m+1)x + 9 = 0$ has equal roots
Show that the roots of the equation $(x-2)(x-3) = k^2$ are always real
For what value of t does $tx^2 + 6x + t = 0$ have equal roots
find nature of roots if t lies between these values ($t \neq 0$)
Find range of values of m for which $5x^2 - 3mx + 5 = 0$ has two real and distinct roots

Tangents to curves

At point where they meet, simultaneous equations
There is only one solution – i.e. equal roots
Discriminant must = 0
For tangency the line meets the curve at only one point

Quadratic Inequalities

To solve a quadratic inequality $f(x) > 0$ or $f(x) < 0$
First find the roots of the equation $f(x) = 0$
Sketch the function $y = f(x)$ marking on the roots
if inequality is $f(x) > 0$ then emphasise curve above axis
if inequality is $f(x) < 0$ then emphasise curve below axis
You can then see range of x which is required
With a quadratic inequality – depending on whether x^2 is positive or negative determines which way up the graph is. You will then want either the region between the roots or outside the roots. depending upon the inequality. Also take care as to whether the inequality is greater than or greater than or equal to etc. as to whether to include the root points.

2 Integration

Differential equations

What is a differential equation

Look at $y = 4x^2$ then dy/dx = 8x

dy/dx = 8x is a differential equation

$y = 4x^2 + c$ is the general solution of the d.e.

$y = 4x^2 – 6$ is a particular solution

Family of curves (graphical example

Simple cases of finding general solution and particular solutions

Integration

$3x^2 - 4x + c$ is called the anti-derivative of 6x – 4

since $d/dx(3x^2 - 4x + c) = 6x – 4$

It comes from 6x – 4 by undoing the differentiation

Leibnitz invented useful notation for anti-derivatives

$$\int 6x - 4 \; dx = 3x^2 - 4x + c$$

In general $\int f(x) \; dx = F(x) + c$ means $F'(x) = f(x)$

Process of calculating an anti-derivative is called integration

The anti-derivative F(x) is called the integral

c is the constant of integration

F(x) is obtained from f(x) by integrating with respect to x

Useful rules

$$\int x^n \; dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\int f(x) + g(x) \; dx = \int f(x) \; dx + \int g(x) \; dx$$

$$\int k \; f(x) \; dx = k \int f(x) \; dx$$ where k is a constant

Methods (Templates)

simple powers of x	x^2
simple powers of x with multipliers	$3x^2$
Simple sums	$3x^2 - 2$
Simple trinomials (quadratics etc)	$5x^2 – 3x + 4$
Brackets squared	$(2x – 1)^2$
Pair of brackets	(2x + 1)(x – 3)
Negative power	x^{-3}
Fractional power	$x^{2/3}$
Negative fractional power	$x^{-1/2}$
Negative power with constant	$2x^{-3}$
Fractional power with constant	$5x^{2/3}$
Negative fractional power with constant	$3x^{-1/2}$
Fraction form with power	$\frac{1}{x^3}$
Constant multipliers	$\frac{6}{x^2}$ *or* $\frac{1}{2x^3}$

Root and further forms (see further in Ex 2B p. 128/9)

NB constant multipliers should be both + and –

Finding the equation y = f(x) from dy/dx and a point on the curve

Rates of Change

Solving Differential equations
Finding general solution
Finding particular solution
Using solution to find a value at a particular time.

Area under a curve

Derivation of formula that: $A(x) = \int f(x)\ dx$

The area between the curve f(x) and the x-axis
bounded by lines x = a and x = b
is given by $\int_a^b f(x)\ dx$

Show by sketches areas associated with a definite integral
Writing down an integral to describe a shaded area

Area under a curve - formula – definite integrals

The area under the curve y = f(x) from x = a and x = b
is given by $\int_a^b f(x)\ dx = \left[F(x)\right]_a^b = \left[F(b)\right] - \left[F(a)\right]$

$\int_a^b f(x)\ dx$ is called a definite integral with lower limit a and upper limit b

Evaluating definite integrals
practice examples

Alternative forms e.g. Find a > 0 if $\int_0^a 2x + 2\ dx = 8$

Area under a curve – calculations

Looking at areas above and below the x axis
Integration for areas below the axis will give negative values for area
(since h > 0 and f(x) < 0)
Integration for areas above the axis will give positive values for area
Since area cannot be negative – ignore the negative sign
The sign merely indicates that the area is below the x axis
Cannot find areas that go above and below the x axis in one integration
Need to break it down into separate sections above and below
then add together

Area between two curves

Take area under lower curve from area under larger curve

$\int_a^b f(x) - g(x)\ dx$

Consider case when part of area is below the x axis
No difference – here the signs take care of themselves

3.1 Calculations in 2D and 3D

Omit this section

3.2 Compound Angle formulae

Related angles

Alternative definitions of sin A, cos A, tan A in terms of x, y, r
sin(-A) = -y/r = - sin A Use also the ASTC diagram
sin (90-A) = x/r = cos A
sin (180 – A) = y/r = sin A
cos(-A) = x/r = cos A
cos (90-A)= y/r = sin A
cos(180-A) = -x/r = -cos A

Sin-cos-tan formula

$$\frac{\sin A}{\cos A} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan A$$

$$\sin^2 A + \cos^2 A = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2 + x^2}{r^2} = 1 \text{ (Pythagoras Theorem)}$$

also reminder π radians = 180°

Some practice reminders
Reminders of exact value table for radians and degrees sin/cos/tan – 30/45/60
p.153 useful examples to practice

Cos(A + B) and cos (A – B)

Proof of Cos (A + B)
To prove for cos (A – B) replace B with –B

Prove simple identities
Using triangles to find expressions for Cos(A+B) using sin A, sin B, cos A, cos B
finding expressions like cos 75 using 30 + 45

sin(A + B) and sin (A – B)

prove using sin(A + B) = cos (90 – (A+B))
Hence result, same for sin(A – B)

Examples showing

simplification	$\sin(\pi + \theta)$
show that	$\sin(\theta + \pi/6)=1/2 (\cos \theta + \sqrt{3} \sin \theta)$
Proving identities	

Sin 2A and cos 2A

Prove by putting B = A
Given sin A find cos A – using a right angled triangle and Pythagoras
Given tan θ find sin 2θ etc
Simplify 2 sin 15 cos 15
More proofs

Solving equations containing sin 2A and cos 2A terms

Replace sin 2A with 2 sin A cos A
    gives common factor solution
Replace cos 2A with 2 $\cos^2$A – 1 if remaining term is cos
Replace cos 2A with 1 - 2 $\sin^2$A if remaining term is sin
    gives a quadratic in sin or cos

Sketching graphs

y = a sin nx

a is amplitude – gives max and minimum values

n gives number of cycles (waveforms) in $0 \le x \le 2\pi$

period is $2\pi/n$

The graph of y = cos (x – π/4)

One cycle of cosine wave has period of 2π

Intersection with x axis when y = 0; cos (x – π/4) = 0

x – π/4 = π/2 or 3π/2 etc

Intersection with y axis when x = 0; i.e. y = cos π/4 = 0.7 approx

Max and min values: 1 when x - π/4 =0 or 2π

i.e. x = π/4 or x = 9π/4

Min value = -1 when x –π/4 = π, 3π etc.

i.e. when x = 5π/4, ….

Note the graph of y = cos (x – π/4) is the same as cos x

but shifted along π/4 to the right (Recall related graphs – f(x - a))

This shift is called the phase angle

In this case the phase angle is π/4

Sketch graphs with different phase angles from equations

Writing down shift and direction from equation

Writing down equation from the graph and noted points

4 The circle

Circle centre (0, 0) and radius r $x^2 + y^2 = r^2$

Write down centre and radius from equation

Write down equation centre (0, 0) and given radius (say 7)

Find the equation of circle, centre O passing through given point (say (3, 4))

Check that a point lies on a circle

Find whether a point lies inside, on or outside a circle (distance formula)

Circle centre (a, b) and radius r $(x-a)^2 + (y-b)^2 = r^2$

Write down centre and radius from equation

Write down equation from iven centre (5, 7) say and given radius (say 3)

Find the equation of circle, centre (2, -1) passing through given point (say (3, 4))

Check that a point lies on a circle

Find whether a point lies inside, on or outside a circle (distance formula)

Right angles in circles – lie on a diameter

Showing an angle is a right angle – use perpendicular gradients

Find equation of a circle passing through 3 points

(2 chords) – perpendicular bisector – intersection – centre

find radius, find equation

If three points form a right angled triangle, then hypotenuse is the diameter

so mid point is the centre, calculate radius

write down equation

Finding if two circles touch – distance between centres = sum of radii

Find if two circles intersect – distance between centres < sum of radii

Find if two circles do not touch – distance between centres > sum of radii

Shortest distance from a point to a circle

lies on line from point to centre – find distance to centre then subtract radius

The general equation of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$

work this through from $(x-a)^2 + (y-b)^2 = r^2$
using numbers first and then with letters
completing the square
Centre is (-g, -f) and radius is $\sqrt{g^2 + f^2 - c}$ provided that $g^2 + f^2 - c > 0$
Note coefficients of x^2 and y^2 must be 1
NB to write down centre
halve the x coeffcient and change the sign
halve the y coefficient and change the sign

Practice
Does a certain equation represent circle
Coefficients of x^2 and y^2 must be 1
$g^2 + f^2 - c > 0$
Given the equation of a circle and a point.
Find the point diametrically opposite
Need centre point
Use triangles
Finding points of intersection of a line and a circle
Simultaneous equations
No real roots – line does not touch circle
Equal roots – line is a tangent to circle
Real and distinct roots – line intersects circle at two points
The shortest distance between two circles (not intersecting)
lies on a line joining their centres
Find distance between two centres and subtract the two radii.
Proving that circles touch externally
Proving that circles touch internally
Finding locus of a point given a condition
Tangents to a circle
Finding equation of a tangent to circle
given equation of circle and point
find centre,
find gradient of line centre to point
find gradient of tangent (perpendicular)
Find equation (point & gradient known)
Find length of the tangent from a point to a given circle
Find centre of circle
Find distance from centre to point
Find radius
Draw a diagram
Use Pythagoras to find length of tangent
Find point of intersection of two tangents to a circle
(equation given and tangency points given)
Find centre
Find gradient of lines joining centre to points
Find gradients of tangents (perpendicular)
Find equations of tangents (point and gradient known)
Find intersection using simultaneous equations
Intersection of lines and circle
Using discriminant
Finding k or c in an equation if a line is a tanent to a circle

Unit 3

1 Vectors

Definition of a vector

magnitude and direction

Definition of a scalar

Examples of each

Directed Line segments

A vector ***u*** can be represented in magnitude and direction

Note printed convention is usually bold type (or underlined or bold italic)

by a directed line segment $\overrightarrow{AB}$

Note that a directed line segment does not indicate position

– only magnitude and direction

Components of a vector in 2 dimensions

displacement in x direction then displacement in y direction

Can be written as a vector $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ i.e. 2 in x direction and 4 in y direction

Draw directed line segments

Write down in vector notation a drawn directed line segment

Find co-ordinates of Q given point P and vector $\overrightarrow{PQ}$

Magnitude of a vector in 2 dimensions

Define magnitude as length and use notation |***u***|

Use Pythagoras

$\underline{u} = \begin{pmatrix} -9 \\ 12 \end{pmatrix} \quad |u|^2 = (-9)^2 + (12)^2 \quad |u|^2 = 225 \quad |u| = 15$

Calculating lengths of vectors - using magnitude formula

Vectors in 3 dimensions

Example with cuboid

Components – x, y and z components

In component form $\overrightarrow{AB} = \begin{pmatrix} x - component \\ y - component \\ z - component \end{pmatrix}$

In general $\overrightarrow{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_b - z_A \end{pmatrix}$ this is called a column vector

Example if P is (6, 2, 0) and Q is (8, -3, -1) then $\overrightarrow{PQ} = \begin{pmatrix} 8-6 \\ -3-2 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$

Magnitude of a 3D vector if $\boldsymbol{u} = \overrightarrow{AB}$ then

$|u| = AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

also known as the distance formula for 3 dimensions

Write down components as a column vector

Calculate length of a vector

If two vectors are equal then corresponding components are equal

Addition and subtraction of vectors

Commonsense solution – consider aeroplane and crosswind

Vectors are added nose to tail

Triangle rule for adding vectors

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ this is called the resultant vector

Simply add components

Finding length of resultant vector as previous

The negative of a vector

Subtracting two vectors

add a negative vector

Multiplying by a number – a scalar

kv has same direction if k positive

kv has opposite direction if k negative

In component form multiply components

Evaluating simple vector expressions 2p + q given p and q

Solving simple vector equations or finding the scalar

Position vectors

Definition

A useful result $\overrightarrow{AB} = \underline{b} - \underline{a}$

Collinear points

Prove three points are collinear

Showing that vectors are scalar multiples of each other

Points dividing lines in given ratios

If one vector is a scalar multiple of another then they are parallel

If they share a common point then the points are collinear

Calculating ratio in which a point divides a line

Position vector of midpoint of AB

Prove formula $m = \frac{1}{2}(a+b)$

Write down co-ordinates of mid-points of lines joing A and B

Points dividing lines in given ratios

e.g. P divides line AB in ratio 3:2 A(2, -3, 4) and B(12, 7, -1)

Find co-ordinates of P

$$\frac{AP}{PB} = \frac{3}{2} \qquad \overrightarrow{AP} = \frac{3}{2}\overrightarrow{PB} \qquad 2\overrightarrow{AP} = 3\overrightarrow{PB}$$

$$2(p-a) = 3(p-b)$$

$$p = \frac{1}{5}(2a+3b) = \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}$$

Deal also with dividing externally

Unit vectors i, j, k

unit vector has magnitude of 1

unit vectors in directions Ox, Oy, Oz denoted by

$$i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad j = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

every vector can be expressed in terms of unit vectors

P(a, b, c) = $a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ = a***i*** + b***j*** + c***k***

The vectors I, j, k form a basis for 3 dimensional space
Write down position vectors of a point A(1, 2, 8)

in form $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and a***i*** + b***j*** + c***k***

Write in component form 2***i*** – 2***k***

Given ***u*** = 2***i*** + 3***j*** + 7***k*** and ***v*** = ***i*** -4***j*** - 2***k***
express u + v and u – v in component form
and find |u + v| etc
Calculating lengths of vectors
Given a vector a***i*** + ½ ***j*** – ½ ***k*** is a unit vector – calculate a
length of vector must be 1
a***i*** + b ***j*** + ½ ***k*** is a unit vector, find relationship between a and b
Prove two vectors are parallel (given in form a***i*** + b ***j*** +c ***k***)
Show that they are scalar multiples
Find the resultant of three vectors acting on a point (add them all up)
Find its magnitude
Given vectors p and q in component form a***i*** + b ***j*** +c ***k***
Find in terms of I, j, k the position vector of the point
which divides PQ in the ratio 2:1

Scalar Product of two vectors
Multiplication of two vectors
Example of occurrence - Force x distance
|f| |x| cos θ
Multiplying two vectors together – product is a real number (scalar)
Hence the name scalar product.
Definition.
The scalar product of two vectors a and b, denoted by a • b is
a • b = |a||b| cos θ (providing neither a nor b is zero)
Component form of a.b
Derive using cosine rule
a.b = $x_1x_2 + y_1y_2 + z_1z_2$
also we have
a.b = |a||b| cos θ
The sign of a.b is determined by the size of θ

0 < θ < 90	a.b > 0
θ = 90	a.b = 0
90 < θ < 180	a.b < 0

θ is the angle between the vectors when they are pointing out from the vertex.
Calculating a.b (Ex 8 p. 209)
Calculating angle between two vectors
Using
a.b = $x_1x_2 + y_1y_2 + z_1z_2$
a.b = |a||b| cos θ

then $\cos\theta = \frac{a \cdot b}{|a||b|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|a||b|}$ providing a, b ≠ 0

Note then that a.b = 0 ↔ cos θ = 0 i.e. θ = 90°
i.e. a is perpendicular to b since we assumed a ≠ 0 and b ≠ 0

Calculating the angle between two vectors
Check the direction of the vectors for θ
Evaluate scalar product
To show that two vectors are perpendicular show that the scalar product is 0
Given two vectors are perpendicular find the value of a coefficient (designated as p)
Some Results using scalar products
$a.a = a^2$
$i^2 = j^2 = k^2 = 1$
i.j = j.k = i.k = 0
Distributive Law a.(b + c) = a.b + a.c
Results from distributive law
if a.(b – c) = 0 then a is perpendicular to b – c

2 Further differentiation and Integration

Derivatives of sin x and cos x
Use the derived function graph to deduce this
Use the definition of the derivative
We must work in radians for this to be true
d/dx(sin x) = cos x d/dx(cos x) = -sin x
Differentiate sin and cos expressions
Find gradient of tangent to a function involving sin and cos
Apply to maximum and minimum values
Stationary points
Stationary values
Differentiation of mixed expressions trig and algebraic (Ex 2B p. 222)
Chain Rule for algebraic functions
Look at patterns – intuitive
Use chain rule notation
dy/dx = dy/du × du/dx
Practice using the chain rule
Chain rule for differentiating trig functions
Use the same way

d/dx sin (…..) = cos(…..) d/dx (…..)
d.dx cos (….) = -sin(….) d/dx (….)

Methods (Templates) also applies to cos x

sin 2x
sin(2x – 3)
sin (x^2 – 1)
√sin x
$\frac{1}{\sin x}$ $\frac{3}{\sin x}$ $\frac{5}{2\sin x}$
$\sin^2$ x, $\sin^4$x
(1 + sin x)^2
√(1 + sin x)
1 – 2$\sin^2$ x 2$\cos^2$x – 1 2 sin x cos x
$\frac{1}{x}-\frac{1}{\sqrt{\sin x}}$

Further Integration

Standard Integral

d/dx$(ax + b)^{n+1}$ = (n + 1)$(ax + b)^n$

so $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a}+c$ where a, b, n are constants, a ≠0 and n ≠ 1

Integration practice of standard functions

bracket will only contain linear functions

Integrals of trig functions

$\int \cos x\, dx = \sin x + c$ $\int \sin x\; dx = -\cos x\; + c$

$\int \cos(ax+b)\; dx = -\frac{\sin(ax+b)}{a}+c$

$\int \sin(ax+b)\; dx = \frac{\cos(ax+b)}{a}+c$

Definite Trig Integrals

Evaluate showing area – working in radians

Calculate areas of shaded regions

Take care with areas below the x axis

Calculate areas between y=cos x and y = sin x between 0 and 5π/4

Split into two sections

3 Exponential and Logarithmic Functions

Growth and decay
Using powers to describe and calculate growth and decay
Writing down growth and decay functions
A special exponential function – the number e
Derivation of where e comes from

limit of $\left(1+\frac{1}{n}\right)^n$ as n tends to infinity

Show computer simulation
limit exists denoted by letter e and is approx: 2.718 281 828 …..

f(x) = e^x is known as the exponential function base e

often denoted as exp (x)

Can you find a number a such that f(x) = a^x

and the derived function f′(x) are the same

The number is in fact a = e

Using e on the calculator

Calculating functions involving e to some powers

(some examples on page 240 Ex 3)

Linking exponential to the log function

$y = a^x \Leftrightarrow x = \log_a y$

or $y = a^x \Leftrightarrow \log_a y = x$

and also $1 = a^0 \Leftrightarrow \log_a 1 = 0$

and $a = a^1 \Leftrightarrow \log_a a = 1$

Writing equations with logs in exponential form

Writing equations with exponentials in log form

Solving log equations by expressing in exponential form

$\log_x 7 = 1$ $\log_2 x = 3$ $\log_4 1 = x$

Finding values of logarithms

$\log_3 27$

Using the log key on your calculator

log is log base 10 or $\log_{10}$

ln is log base e or $\log_e$ known as natural logs

Do not confuse them

Solve $\log_{10} x = 2$

Rules of logarithms

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^p = p \log_a x$$

Prove these from rules of indices – see proof on page 242

Simplify logs (all with same base) log 7 + log 2

Simplify logs with given base and evaluate $\log_4 18 - \log_4 9$

see page 242 for examples of these

Solving log equations page 243

Solving exponential equations

Solve $5^x = 4$ – take log base 10 of both sides

Example with compound interest formula

$A = PR^n$ say $1.08^n = 2.5$ n = 11.91 (again using log base 10)

When growth or decay base is e

then it is simpler to take logs to the base e of both sides

e.g. $e^{1.24t} = 125$ $1.24t \log_e e = \log_e 125$

since $\log_e e$ is 1 then $t = \log_e 125 / 1.24$ t = 3.89

Experiment & Theory

$y = ax^n$ and $y = a\,b^x$ could both model similar graphs

Can solve dilemna by taking logs

$$y = ax^n$$

$$\log y = \log ax^n = \log a + \log x^n$$

$$\log y = \log a + n \log x$$

$$\log y = n \log x + \log a$$

This is like Y = mX + c where Y = log y and X = log x

For $y = a\,b^x$

$$y = ab^x$$

$$\log y = \log ab^x$$

$$\log y = \log a + \log b^x$$

$$\log y = \log a + x \log b$$

$$\log y = x \log b + \log a$$

This is like Y = mx + c where Y = log y

The graph of each is a straight line, by drawing the line,
m and c can be found

A plot of log y against log x will result in an equation of the form $y = ax^n$

A plot of log y against x will result in an equation of the form $y = a\,b^x$

Some examples with graphs

4. The Wave function a cos x + b sin x

Investigating the graph $y = \cos x + \sin x$ for $0 \le x \le 2\pi$

Use graph program

Graph is like $y = \cos x$ but amplitude is $\sqrt{2}$ and it is displaced $\pi/4$ to the right

i.e. $y = \sqrt{2}\cos(x - \pi/4)$

Check using compound angle formula that

$$\mathbf{\sqrt{2}\cos(x - \pi/4) = \cos x + \sin x}$$

Expressing a cos x + b sin x in the form $R\cos(x \pm \alpha)$

often the greek letter α is used

Reminder of compound angle formula

sin (A ± B) and cos (A ± B)

Proof:

Let $a\,\underline{\cos x} + b\,\underline{\sin x} = R(\cos(x - \alpha) = R\,\underline{\cos x}\cos\alpha + R\,\underline{\sin x}\sin\alpha$

Equating coefficients of cos x and sin x we find:

$R\cos\alpha = a$

$R\sin\alpha = b$

Squaring and adding

$$R^2\cos^2\alpha \;+\; R^2\sin^2\alpha \;=\; a^2 + b^2$$

$$R^2\left(\cos^2\alpha + \sin^2\alpha\right) = a^2 + b^2$$

$$R^2 = a^2 + b^2$$

$$R = \sqrt{a^2 + b^2}$$

Dividing (we want sin divided by cos to get tan)

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a}$$

$$\tan\alpha = \frac{b}{a}$$

The quadrant of a is found from the signs of cos α and sin α

So a cos x + b sin x = R cos (x - α) where $R = \sqrt{a^2 + b^2}$ **and** $\tan\alpha = \dfrac{b}{a}$

In practice do not use the formula

since there are 4 possible forms we could use.

$R\cos(x \pm \alpha)$ and $R\sin(x \pm \alpha)$

Work it out for each one in turn from first principles

Examples of use

Maxima and minima of a cos x + b sin x

Finding max and min and corresponding value for x

Given in form $y = 2\sin(x + 75)$

Given in form 12 cos x + 5 sin x

More involved form where $f(x) = 1 + \sqrt{2}\cos x - \sqrt{2}\sin x$

Solving Trig equations using the wave function

Solve $\sin 2x + 3\cos 2x + 1 = 0$ for $0 \le x \le 180°$

Express in form $R\cos(2x - \alpha)$

which gives $\sqrt{10}\cos(2x - 18°) = -1$

x = 63 or 135 to nearest degree

You will be told which form to use in the question.